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On hydrostatic flows in isentropic coordinates

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The hydrostatic primitive equations of motion which have been used in large-scale weather prediction and climate modelling over the last few decades are analysed with variational methods in an isentropic Eulerian framework. The use of material isentropic coordinates for the Eulerian hydrostatic equations is known to have distinct conceptual advantages since fluid motion is, under inviscid and statically stable circumstances, confined to take place on quasi-horizontal isentropic surfaces. First, an Eulerian isentropic Hamilton's principle, expressed in terms of fluid parcel variables, is therefore derived by transformation of a Lagrangian Hamilton's principle to an Eulerian one. This Eulerian principle explicitly describes the boundary dynamics of the time-dependent domain in terms of advection of boundary isentropes s_B; these are the values the isentropes have at their intersection with the (lower) boundary. A partial Legendre transform for only the interior variables yields an Eulerian 'action' principle. Secondly, Noether's theorem is used to derive energy and potential vorticity conservation from the Eulerian Hamilton's principle. Thirdly, these conservation laws are used to derive a wave-activity invariant which is second-order in terms of small-amplitude disturbances relative to a resting or moving basic state. Linear stability criteria are derived but only for resting basic states. In mid-latitudes a time-scale separation between gravity and vortical modes occurs. Finally, this time-scale separation suggests that conservative geostrophic and ageostrophic approximations can be made to the Eulerian action principle for hydrostatic flows. Approximations to Eulerian variational principles may be more advantageous than approximations to Lagrangian ones because non-dimensionalization and scaling tend to be based on Eulerian estimates of the characteristic scales involved. These approximations to the stratified hydrostatic formulation extend previous approximations to the shallow-water equations. An explicit variational derivation is given of an isentropic version of Hoskins & Bretherton's model for atmospheric fronts.

1. Introduction

My original intention was to analyse atmospheric fronts in an isentropic version of Hoskins & Bretherton's (1972) geostrophic momentum model by using conservation properties. The idea was to consider fronts as local discontinuities where certain conservation laws were broken and other ones not. It turned out that a variational formulation, from which conservation laws would arise, had neither been derived for hydrostatic flows in isentropic coordinates nor for the geostrophic momentum approximation. In these flows isentropes generally intersect the boundary, and when the flow is expressed in isentropic coordinates the associated time-dependent boundary conditions are seldom formulated explicitly. Moreover, variational and Hamiltonian

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formulations of isentropic hydrostatic flows have not been recorded. Roulstone & Brice (1995) state a variational formulation of quasi-hydrostatic flows but with no reference to time-dependent boundary conditions nor further analysis of stability and conservation properties of the hydrostatic system. In this paper, I will therefore systematically derive isentropic Eulerian variational principles and analyse stability and conservation properties of the isentropic hydrostatic primitive equations with explicit inclusion of the time-dependent boundary dynamics. An isentropic version of Hoskins & Bretherton's model will finally be derived in the discussion from such an Eulerian isentropic variational principle.

What are hydrostatically balanced flows? The hydrostatic primitive equations of motion have been used in large-scale numerical weather prediction and in climate modelling for the last few decades. While the fully compressible equations of motion have a vertical momentum equation of the form

$$\frac{\mathrm{D}w}{\mathrm{D}t} \equiv \frac{\partial w}{\partial t} + (\boldsymbol{u}\cdot\nabla_3)\, w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{1.1}$$

the hydrostatic equations have one of the form

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{1.2}$$

where vertical coordinate z is part of a local Cartesian coordinate frame with z normal to the Earth's surface, w is the vertical velocity, $\boldsymbol{u}$ and ∇_3 are the three-dimensional velocity and gradient vector, respectively, ρ is the density, p the pressure and g the local gravitational acceleration. Hydrostatically balanced flows are based on the approximation that horizontal length and velocity scales exceed vertical ones, with inverse aspect ratio $1/\delta$. Hydrostatic balance is thus valid to leading order in δ for large-scale flows in the atmosphere and oceans: these flows are confined to a shell thin compared with their large horizontal scales.

When and why are isentropic coordinates of interest? Static stability for compressible flows has the form

$$N^2 \equiv \frac{g}{c_p}\frac{\partial s}{\partial z} > 0 \tag{1.3}$$

with N^2 the Brunt–Väisälä frequency, c_p the specific heat at constant pressure, and s the entropy. When condition (1.3) holds, we can transform from a Cartesian coordinate system, x, y, z, to an isentropic one, x, y, s. Since entropy s is materially advected by the three-dimensional velocity, both in the non-hydrostatic and hydrostatic systems, fixed boundaries in Cartesian space generally become time-dependent boundaries in isentropic space. Nevertheless, the hydrostatic dynamics in the interior, away from boundaries, simplifies in the inviscid and statically stable case to quasi-horizontal dynamics which takes place on the isentropic coordinate surfaces. Only the horizontal velocity then enters the equations of motion, while reference to the constrained vertical velocity vanishes. (Vertical advective velocity w is constrained in the sense that vertical fluid parcel acceleration $\mathrm{D}w/\mathrm{D}t$ is absent in the vertical hydrostatic momentum balance.) Friction and forcing are accordingly associated with transport of momentum and mass across isentropic surfaces. This conceptually simpler picture of predominantly two-dimensional large-scale atmospheric motion on isentropic surfaces with weak cross-isentropic flows directly associated with friction and forcing has been very illuminating in explaining atmospheric circulation patterns (e.g. Hoskins, McIntyre & Robertson 1985; Hoskins 1991).

The paper is organized as follows. The variational formulation of hydrostatic flows is considered as the basic or 'parent' dynamics (McIntyre & Roulstone 1996). Readers who are not familiar with variational and Hamiltonian formulation in fluid dynamics are referred to review papers by Salmon (1988*a*), Shepherd (1990), and Morrison (1998). In § 2, an Eulerian isentropic Hamilton's principle for hydrostatic compressible flows is derived by transformation of a Lagrangian Hamilton's principle for hydrostatic compressible flows to an Eulerian one. While in a Lagrangian Hamilton's principle the positions of fluid parcels are the dependent variables, in the Eulerian Hamilton's principle the fluid parcels are the dependent variables as function of horizontal spatial coordinates, entropy and time. This Lagrangian Hamilton's principle for hydrostatic flows follows immediately from Hamilton's principle (e.g. Salmon 1988*a* and Morrison 1998) for non-hydrostatic flows by neglecting the contribution $\frac{1}{2}w^2$ from the kinetic energy. Alternatively, the Lagrangian Hamilton's principle for hydrostatic flows can be derived using Dirac's constrained Hamiltonian theory (Bokhove 1996, Chap. 5; and Bokhove 1999, § 3.2; see also Theiss 1997). A partial Legendre transform of the Eulerian Hamilton's principle for only the interior variables yields an Eulerian 'action' principle. Noether's theorem is used to derive energy and potential vorticity conservation laws from the Eulerian Hamilton's principle. These conservation laws are subsequently used in § 3 to derive a wave-activity invariant which is second order in terms of small-amplitude disturbances relative to a resting or moving basic state. Linear stability criteria are derived but only for resting basic states. In mid-latitudes a time-scale separation between gravity and vortical modes occurs. This separation motivates a discussion in § 4 of derivations of conservative geostrophic and ageostrophic approximations to the Eulerian action principle for hydrostatic flows. Such derivations of approximate models from the stratified hydrostatic equations extend previous derivations (e.g. Salmon 1985; Allen & Holm 1996; McIntyre & Roulstone 1996) of approximate conservative models from the shallow-water equations. Throughout the article, the close correspondence between shallow-water equations and hydrostatic equations, when the latter are expressed in isentropic coordinates, is emphasized. The three-dimensional calculations in this paper are greatly simplified when the dynamics is only considered in a two-dimensional vertical plane. Corresponding two-dimensional calculations may be used as a first check of the results.

2. Eulerian variational principles in isentropic coordinates

2.1. *Eulerian Hamilton's principle*

Consider a Lagrangian variational principle for hydrostatic compressible flows

$$0 = \delta S_c[\xi_h] = \delta \int_{\tau_0}^{\tau_1} \mathrm{d}\,\tau \int_D \mathrm{d}\boldsymbol{a}\,\rho_0(\boldsymbol{a}) \left\{ \left[\frac{1}{2}\frac{\partial \xi_h}{\partial \tau} + R_h(\xi_1,\xi_2) \right] \frac{\partial \xi_h}{\partial \tau} - U\big(s,\rho\big) - g\,\xi_3 \right\}, \quad (2.1)$$

in which fluid parcel positions $\boldsymbol{\xi}(\boldsymbol{a},\tau) = (\xi_1,\xi_2,\xi_3)^T$ are the dependent variables as functions of a continuum of fluid-parcel labels $\boldsymbol{a} = (a,b,c)^T$ and time τ, and in which $h = 1,2$. The Lagrangian density in (2.1) is the kinetic energy, modified by the Coriolis effect, minus the internal and potential energy. The horizontal components of the velocity are $\partial\xi_{1,2}(\boldsymbol{a},\tau)/\partial\tau$. The Coriolis parameter $f(\xi_1,\xi_2) = \hat{\boldsymbol{z}}\cdot\nabla_3\times\boldsymbol{R}$ with $\hat{\boldsymbol{z}}$ the unit vector in the vertical and with $R_3 = 0$. The internal energy $U(s,\rho)$ is a function of entropy s, which is conserved on a fluid parcel, and density ρ. The reference density $\rho_0(\boldsymbol{a})$ may be chosen such that $\rho(\boldsymbol{a},\tau=0) \equiv \rho_0(\boldsymbol{a})$. An element of mass $\mathrm{d}m$ is thus defined by

$$\mathrm{d}m = \rho_0(\boldsymbol{a})\,\mathrm{d}\boldsymbol{a} = \rho(\boldsymbol{\xi}, t)\,\mathrm{d}\boldsymbol{\xi} \tag{2.2}$$

with time denoted by t in an Eulerian framework. The first step in a variation of internal energy U is obtained from the first law of thermodynamics

$$\mathrm{d}U = T\,\mathrm{d}s - p\,\mathrm{d}\big(1/\rho\big), \tag{2.3}$$

with T the temperature and p the pressure. Pressure p is via an equation of state

$$p = p(s, \rho) \tag{2.4}$$

related to entropy and density. Hamilton's principle (2.1) follows immediately from Hamilton's principle for non-hydrostatic flows (e.g. equation (2.6) in Salmon 1988*a* and equation (104) in Morrison 1998) by an aspect-ratio trunction of the vertical contribution $(\partial\xi_3/\partial\tau)^2$ to the kinetic energy.

Variation of (2.1) with respect to $\delta\xi_h$, with endpoint conditions $\delta\xi_i(\boldsymbol{a}, \tau_0) = \delta\xi_i(\boldsymbol{a}, \tau_1) = 0$ $(i = 1, 2, 3)$, thermodynamic relations (2.3) and (2.4), initial conditions $\boldsymbol{\xi}(\boldsymbol{a}, 0) \equiv \boldsymbol{a}$, and suitable boundary conditions yields the hydrostatic equations of motion in the form

$$\rho_0(\boldsymbol{a})\left[\frac{\partial^2\xi_h}{\partial\tau^2} + \left(\frac{\partial R_h}{\partial\xi_j} - \frac{\partial R_j}{\partial\xi_h}\right)\frac{\partial\xi_j}{\partial\tau}\right] = -A_{hj}\,\frac{\partial p}{\partial a_j}, \tag{2.5}$$

$$0 = -A_{3j}\,\frac{\partial p}{\partial a_j} - g\,\rho_0(\boldsymbol{a}), \tag{2.6}$$

where

$$A_{pj} = \tfrac{1}{2}\,\epsilon_{pqr}\,\epsilon_{jkl}\,\frac{\partial\xi_q}{\partial a_k}\,\frac{\partial\xi_r}{\partial a_l} \tag{2.7}$$

with permutation symbol ϵ_{pqr} and $p, q, r = 1, 2, 3$ $(h = 1, 2)$. Equation (2.6) is the hydrostatic balance condition (1.2) in Lagrangian form. The system (2.5)–(2.6) along with thermodynamic relations is closed under suitable initial and boundary conditions because the positions $\xi_3(\boldsymbol{a}, \tau)$ may be obtained from the partial differential equation (2.6): by differentiating (2.6) with respect to time τ and by using the boundary conditions to eliminate the terms $\partial\xi_3/\partial\tau$ at the boundary the appropriate boundary conditions for ξ_3 can be found. Details on the variation of (2.1), or of similar principles, can be found in Salmon (1988*a*), Morrison (1998) and Bokhove (1999).

The Lagrangian Hamilton's principle (2.1) for hydrostatic balanced flows may be transformed into an isentropic Eulerian Hamilton's principle by a coordinate change from label coordinates a, b, $c = s$ and time τ to isentropic Eulerian coordinates x, y, entropy s and time $t \equiv \tau$ and a transformation from Lagrangian variables $\boldsymbol{x}(\boldsymbol{a}, s, \tau)$ to Eulerian variables $\boldsymbol{a}(\boldsymbol{x}, s, t) = (a(\boldsymbol{x}, s, t), b(\boldsymbol{x}, s, t))^T$. Consider the product of Jacobians $\partial(x, y, t, s)/\partial(a, b, \tau, s)$ and $\partial(a, b, \tau, s)/\partial(x, y, t, s)$, i.e.

$$\begin{pmatrix} x_a & x_b & x_\tau & x_s \\ y_a & y_b & y_\tau & y_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_x & a_y & a_t & a_s \\ b_x & b_y & b_t & b_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

where subscripts denote partial derivatives. From (2.8) one deduces that

$$\boldsymbol{v} \equiv \frac{\partial\boldsymbol{x}}{\partial\tau} = -\boldsymbol{\Gamma}^{-1}\,\frac{\partial\boldsymbol{a}}{\partial t} \Longleftrightarrow \frac{\partial a^i}{\partial t} + u^k\,\frac{\partial a^i}{\partial x^k} = 0, \tag{2.9}$$

which amounts physically to advection of fluid parcel labelled $\boldsymbol{a}$ by horizontal fluid

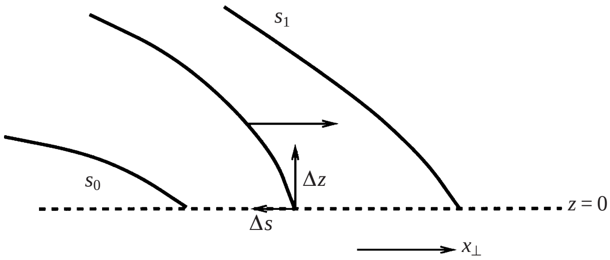

FIGURE 1. Sketch of the relation between geopotential $\phi \equiv g\,z$ and entropy s variations at the Earth's surface $z = 0$. $s_{0,1}$ with $s_0 < s_1$ are two values of an isentropic surface. The horizontal component normal to entropy contours at the lower boundary is considered. The resulting relations for other geometries follow likewise.

velocity $\boldsymbol{v} \equiv (u, v)^T$ on isentropic surfaces labelled by coordinate s. The tensor $\boldsymbol{\Gamma}$ is defined by $\Gamma^i_k = \partial a^i/\partial x^k$. All indices range from $i = 1, 2$ and care is taken in their placement. Hence, one finds

$$0 = \delta S_c[\boldsymbol{a}] = \delta \int_{t_0}^{t_1} \mathrm{d}t\, \mathscr{L}_c = \delta \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B(\boldsymbol{x},t)}^{\infty} \mathrm{d}s\, \sigma(\boldsymbol{x}, s, t)$$
$$\times \left\{ \left(\tfrac{1}{2}\, u_h(\boldsymbol{x}, s, t) + R_h(\boldsymbol{x}) \right) u^h(\boldsymbol{x}, s, t) - U\big(s, \rho(\boldsymbol{x}, s, t)\big) - g\, z(\boldsymbol{x}, s, t) \right\} \qquad (2.10)$$

as the Eulerian Hamilton's principle, in which velocity u^h is the shorthand defined in (2.9), and in which pseudodensity

$$\sigma(\boldsymbol{x}, s, t) \equiv \sigma_0(\boldsymbol{a}, s)\, J(a, b) = -\frac{1}{g}\, \frac{\partial p(\boldsymbol{x}, s, t)}{\partial s} = \rho(\boldsymbol{x}, s, t)\, \frac{\partial z(\boldsymbol{x}, s, t)}{\partial s} \qquad (2.11)$$

with the horizontal Jacobian $J(a, b) \equiv \partial_x a\, \partial_y b - \partial_x b\, \partial_y a$ appears, where $u_h = \delta_{hj}\, u^j$ (∂_{hj} is the Kronecker-delta symbol), and in which variations are taken with respect to fluid parcel variables $\boldsymbol{a}$. $\mathscr{L}_c$ is a Lagrangian functional for hydrostatic flow. The domain is for example a horizontally infinite, closed or periodic domain, above a mountain range h_B, i.e. $z > h_B(\boldsymbol{x})$. In isentropic coordinates we have $s > s_B(\boldsymbol{x}, t)$ and the maximum horizontal extent of the domain is D_H.

In the evaluation of (2.10) the following expressions, or variations or time-derivatives thereof, are useful:

$$(\Gamma^{-1})^i_k\, \Gamma^k_j = \delta^i_j, \qquad (2.12)$$

$$\frac{\partial \left(\sigma (\Gamma^{-1})^m_k \right)}{\partial t} + \frac{\partial \left(\sigma u^j\, (\Gamma^{-1})^m_k \right)}{\partial x^j} = \sigma\, (\Gamma^{-1})^n_k\, \frac{\partial u^m}{\partial x^n}, \qquad (2.13)$$

$$-\int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s\, \sigma\, \delta\big(U + g\, z\big) = -\int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s\, \frac{p}{\rho}\, \delta\sigma + \int_{D_H} \mathrm{d}\boldsymbol{x} \left(\sigma\, \frac{p}{\rho} \right) \bigg|_{s_B} \delta s_B. \quad (2.14)$$

Subscripts B imply evaluation at the boundary B at $z = h_B(\boldsymbol{x})$.

The continuity equation expressed in terms of the pseudodensity appears directly from definitions (2.9) and (2.11)

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \sigma_0}{\partial a^k}\, \frac{\partial a^k}{\partial t}\, J(a, b) + \sigma_0\, \epsilon^{ij}\, \epsilon_{mn} \left(\frac{\partial^2 a^m}{\partial x^i \partial t} \right) \frac{\partial a^n}{\partial x^j} = -u^j\, \frac{\partial \sigma}{\partial x^j} - \sigma\, \frac{\partial u^j}{\partial x^j}. \qquad (2.15)$$

The variation of (2.10) with respect to δa^k and $(\delta a^k)_B$ yields

$$0 = \delta S_c[\boldsymbol{a}] = \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s\, \sigma \left(\Gamma^{-1}\right)_k^n \left\{ \frac{\partial u_n}{\partial t} + u^j \frac{\partial u_n}{\partial x^j} + u^j \frac{\partial R_n}{\partial x^j} - u^j \frac{\partial R_j}{\partial x^n} \right.$$
$$\left. + \frac{\partial M}{\partial x^n} \right\} \delta a^k + \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \left\{ \left(\left(\Gamma^{-1}\right)_k^n \frac{\partial s_B}{\partial x^n} \left(\delta a^k\right)_B - \delta s_B \right) B_M\, \sigma \right.$$
$$\left. - p \left((\delta\, z)_B + \frac{\partial z}{\partial s} \delta s_B \right) - \sigma \left(\Gamma^{-1}\right)_k^n (u_n + R_n) \left(\frac{\partial s_B}{\partial t} + u^j \frac{\partial s_B}{\partial x^j} \right) \left(\delta a^k\right)_B \right\} \Bigg|_{s_B}, \quad (2.16)$$

where a function

$$B_M \equiv \left(\tfrac{1}{2}\, u_h + R_h\right) u^h - M \quad (2.17)$$

has been defined with Montgomery potential

$$M \equiv E + g\, z \quad (2.18)$$

and enthalpy $E \equiv U + p/\rho$. The first two boundary terms in (2.16) cancel one another with the help of the following relations:

$$\delta s_B = \frac{\partial s_B}{\partial \left(a_B^m\right)}\, \delta \left(a_B^m\right) \quad \text{and} \quad \delta \left(a_B^m\right) = \left(\delta\, a^m\right)_B + \left(\frac{\partial a^m}{\partial s} \right)_B \delta s_B, \quad (2.19)$$

in which the variation of $a_B^m \equiv a^m\left(\boldsymbol{x}_h, s = s_B(\boldsymbol{x}, t), t\right)$ is not equal to the boundary value of the variation of $a^m(\boldsymbol{x}, s, t)$. The third and fourth boundary terms vanish when we use the relation

$$0 = \delta \left(z\, \big|_{z=h_B}\right) = \frac{\partial z(\boldsymbol{x}, s, t)}{\partial s} \bigg|_{z_B} \delta s_B + (\delta z(\boldsymbol{x}, s, t)) \bigg|_{s_B}; \quad (2.20)$$

the first equality in (2.20) emerges since $z(\boldsymbol{x}, s_B(\boldsymbol{x}, t)) = h_B(\boldsymbol{x})$ and hence

$$\delta\, z(\boldsymbol{x}, s_B(\boldsymbol{x}, t)) = \delta\, h_B(\boldsymbol{x}) = 0$$

and the second one in (2.20) follows geometrically from figure 1.

The equations of motion which arise from these variations are thus horizontal advection of the boundary entropy at $z = h_B(\boldsymbol{x})$

$$\left(\delta a^k\right)_B : \quad \frac{\partial s_B}{\partial t} + u^k \frac{\partial s_B}{\partial x^k} = 0 \quad (2.21)$$

and the horizontal momentum equations

$$\left(\delta a^k\right) : \quad \frac{\partial u_m}{\partial t} + u^k \frac{\partial u_m}{\partial x^k} + u^k \left(\frac{\partial R_m}{\partial x^k} - \frac{\partial R_k}{\partial x^m} \right) + \frac{\partial M}{\partial x^m} = 0. \quad (2.22)$$

Since u_m is defined by (2.9) the momentum equations are second-order partial differential equations in time for the fluid labels. It may come as a surprise that the advection of boundary entropy does not involve the mountain h_B. However, at the boundary one finds in isentropic coordinates that

$$\frac{\mathrm{D}(z - h_B)}{\mathrm{D}t} = -\left(\frac{\partial z}{\partial s} \right) \bigg|_{s_B} \frac{\mathrm{D}s_B}{\mathrm{D}t} \bigg|_{s_B} + (\boldsymbol{v} \cdot \nabla|_s\, z(x, y, s))_{s_B} - (\boldsymbol{v} \cdot \nabla|_s\, h_B(\boldsymbol{x}))_{s_B} = 0$$
$$\Longrightarrow \frac{\mathrm{D}s_B}{\mathrm{D}t} \bigg|_{s_B} = 0, \quad (2.23)$$

while in Cartesian coordinates one finds that

$$\left(\frac{\partial s(\boldsymbol{x},z,t)}{\partial t} + (\boldsymbol{v}\cdot\nabla)\, s(\boldsymbol{x},z,t) + (\boldsymbol{v}\cdot\nabla)h_B(\boldsymbol{x})\,\frac{\partial s(\boldsymbol{x},z,t)}{\partial z}\right)_{z=h_B} = 0. \qquad (2.24)$$

Equations of motion (2.21) and (2.22) need to be complemented with the first law of thermodynamics (2.3) and the definition of the pseudodensity (2.11). The latter two with (2.18) imply that

$$T = \frac{\partial M}{\partial s}. \qquad (2.25)$$

With the ideal gas law $p = \rho\, R\, T$, in which $R = c_p - c_v$ is the gas constant and c_v the specific heat at constant volume, equations (2.3), (2.11) and (2.25) can be reduced to the elliptic equation

$$\sigma = -\frac{p_{00}}{g}\frac{\partial}{\partial s}\left[\left(\frac{1}{T_{00}}\frac{\partial M}{\partial s}\right)^{c_p/R} \mathrm{e}^{-(s-s_{00})/R}\right] \qquad (2.26)$$

with reference temperature T_{00}, reference pressure p_{00} and reference entropy s_{00}. The lower boundary condition at $s = s_B(\boldsymbol{x},t)$ is

$$M = c_p\,\frac{\partial M}{\partial s} + g\, h_B(\boldsymbol{x}) \qquad (2.27)$$

and the upper one at $z, s \to \infty$ is

$$p \equiv p_{00}\left(M_s/T_{00}\right)^{c_p/R} \mathrm{e}^{-(s-s_{00})/R} \to 0. \qquad (2.28)$$

Alternatively, an upper stratospheric boundary condition of prescribed pressure at $s = s_U(\boldsymbol{x},t)$ may be specified as

$$p \equiv p_{00}\left(M_s/T_{00}\right)^{c_p/R} \mathrm{e}^{-(s-s_{00})/R} \equiv p_U(\boldsymbol{x}). \qquad (2.29)$$

With the ideal gas law (2.18) becomes $M = c_p\, T + g\, z$.

2.2. *Eulerian action principle*

The generalized interior momentum corresponding to (2.1) is (e.g. Morrison 1998 and Bokhove 1999)

$$\begin{aligned}\pi^*_k(\boldsymbol{x},t) = \delta\mathscr{L}[\boldsymbol{a}]/\delta\left(\frac{\partial a^k}{\partial t}\right) &= \sigma\,(\Gamma^{-1})_k^m\left(\delta_{mn}\,(\Gamma^{-1})_j^n\,\frac{\partial a^j}{\partial t} - R_m\right)\\ &= -\sigma\,(\Gamma^{-1})_k^m\,(u_m + R_m).\end{aligned} \qquad (2.30)$$

An Eulerian action principle follows after a Legendre transform and may be rewritten in terms of $\boldsymbol{\pi}^*$ and $\boldsymbol{a}$ or in terms of $\boldsymbol{v}$ and $\boldsymbol{a}$. One finds either

$$0 = \delta\int_{t_0}^{t_1}\mathrm{d}t\,\mathscr{L}_c = \delta\int_{t_0}^{t_1}\mathrm{d}t\left\{\int_{D_H}\mathrm{d}\boldsymbol{x}\int_{s_B}^{\infty}\mathrm{d}s\,\pi^*_k\,\frac{\partial a^k}{\partial t} - \mathscr{H}[\pi^*_i, a^i]\right\} \qquad (2.31)$$

or

$$0 = \delta\int_{t_0}^{t_1}\mathrm{d}t\left\{\int_{D_H}\mathrm{d}\boldsymbol{x}\int_{s_B}^{\infty}\mathrm{d}s\left[-\sigma\left(u_m + R_m\right)\left(\Gamma^{-1}\right)_k^m\frac{\partial a^k}{\partial t}\right] - \mathscr{H}[u_i, a^i]\right\} \qquad (2.32)$$

with the Hamiltonian as Legendre transform

$$\mathscr{H}[\pi^*_i, a^i] = \mathscr{H}[u_i, a^i] = \int_{D_H}\mathrm{d}\boldsymbol{x}\int_{s_B}^{\infty}\mathrm{d}s\,\sigma\,\left\{\tfrac{1}{2}\,u_i\,u^i + U(s,\rho) + g\,z\right\}. \qquad (2.33)$$

Independent variations in (2.31) or (2.32) are taken with respect to $\boldsymbol{\pi}^*, \boldsymbol{a}$ or $\boldsymbol{v}, \boldsymbol{a}$, respectively. Similar calculations to those for variations of Hamilton's principle (2.1) yield the horizontal momentum equations (2.22) on isentropic surfaces for variations $\delta\boldsymbol{a}$, (2.9) for variations $\delta\boldsymbol{v}$, and again

$$\left[\sigma\,(\Gamma^{-1})^n_k\,(u_n+R_n)\left(\frac{\partial s_B}{\partial t}+\boldsymbol{v}\cdot\nabla s_B\right)\right]\Big|_{s=s_B}=0 \tag{2.34}$$

for $(\delta a^k)_B$. The momentum corresponding to $\partial a^k_B/\partial t$ is zero which signals that the Lagrangian (2.1) is singular but only at the boundary. Dirac's theory for singular Lagrangians (Dirac 1964) may be applied but I am not pursuing this route.

2.3. *Hamiltonian formulation*

A Hamiltonian formulation of the hydrostatic primitive equations in isentropic coordinates, in terms of label variables $\boldsymbol{a}$ and velocity $\boldsymbol{v}$, may be derived directly from action principle (2.32) (Sudarshan & Mukunda 1974, p. 132 ff. and p. 422 ff.) – for the moment ignoring boundary contributions by taking s_B constant. Subsequently, a reduced Hamiltonian formulation can, in principle, be found either from the Lagrangian or Eulerian Hamiltonian fluid-parcel formulations as a corollary of the reduction theory developed by Marsden & Weinstein (1983) (see also Mazer & Ratiu 1989 and Morrison 1998).

2.4. *Noether's theorem and Casimir invariants*

Noether's (1918) first theorem may be applied for point transformations which leave the Lagrangian density in the action (2.10) invariant. Since this action is expressed in isentropic Eulerian coordinates, the rôle of independent and dependent variables has changed relative to the same action S_c expressed in the Lagrangian fluid parcel coordinates. Following notation of Padheye & Morrison (1996) who consider Noether's theorem for a Lagrangian Hamilton's principle for non-hydrostatic, compressible flows, we find

$$S_k\,\delta a^k+\partial_i\delta\Upsilon^i=0 \tag{2.35}$$

where $k=1,2$; $\boldsymbol{x}\equiv(t,x,y,s)^T$, $\boldsymbol{a}=(a^1,a^2)^T$; $\partial_o=\partial/\partial t$; $\nabla=(\partial_x,\partial_y)^T$; $\partial_3=\partial_s=\partial/\partial s$; and

$$\left.\begin{aligned} S_k\equiv\frac{\delta A}{\delta a^k}=\frac{\partial L}{\partial a^k}-\partial_j\,\frac{\partial L}{\partial(\partial_j\,a^k)},\quad \delta\Upsilon^i\equiv L\,\delta x^i+\frac{\partial L}{\partial(\partial_i a^k)}\,\delta a^k+\Lambda^i,\\ \delta a^k=\Delta a^k(\boldsymbol{x},\boldsymbol{a})-\partial_j a^k\,\delta\boldsymbol{x}^j\end{aligned}\right\} \tag{2.36}$$

with $i,j=0,1,2,3$. Noether's first theorem now reads

$$\partial_j(\delta\Upsilon^j)=\frac{\partial\delta\Upsilon^0}{\partial t}+\nabla\cdot(\delta\Upsilon)+\frac{\partial\delta\Upsilon^s}{\partial s}=0 \tag{2.37}$$

with $\Upsilon^s\equiv\Upsilon^l$ for $l=3$. Understanding the origin of symmetries and associated conservation laws from an Eulerian viewpoint is of interest in the search of both symmetry-preserving approximations to Eulerian variational principles and symmetry-preserving Eulerian discretizations. Direct application of Noether's first theorem is cumbersome due to the time-dependent domain in isentropic coordinates. The derivation of conservation laws directly from the variational principle for each separate symmetry is more straightforward. Some detail of this derivation is provided to highlight subtleties associated with the time-dependent domain. Although Eulerian variational principles in terms of fluid parcel variables are less well-known, they generally have the same validity as Lagrangian variational principles. Approximations to Eulerian variational principles may nevertheless be more advantageous because non-dimensionalization and scaling tend to be based on Eulerian estimates of the characteristic scales involved – examples supporting this assertion are given in § 4.

First, let us consider the same particle relabelling transformation as in Lagrangian coordinates (e.g. Salmon 1983, 1988*a* and Padheye & Morrisson 1996) but now expressed in isentropic, Eulerian coordinates, i.e. consider point transformations of the form

$$\hat{\mathsf{x}} = \mathsf{x}, \quad \hat{a}(\hat{\mathsf{x}}, a) = a(\mathsf{x}) + \Delta a(\mathsf{x}, a). \tag{2.38}$$

Hence $\delta a(\mathsf{x}, a) \equiv \hat{a}(\hat{\mathsf{x}}, a) - a(\mathsf{x}) = \Delta a(\mathsf{x}, a)$. We seek transformations (2.38) for which the Euler–Lagrange equations remain invariant, i.e. $\hat{S}_c[\hat{a}] = S_c[\hat{a}]$. Together with covariance of the action $\hat{S}_c[\hat{a}] = S_c[a]$, these transformations have to satisfy

$$\int_D \mathrm{d}\mathsf{x} \left(L(a, \partial a, \mathsf{x}) - \frac{\partial(\hat{\mathsf{x}})}{\partial(\mathsf{x})} L(\hat{a}, \hat{\partial}\hat{a}, \hat{\mathsf{x}}) \right) = 0 \tag{2.39}$$

with $\int \mathrm{d}\mathsf{x} = \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_{min}} \mathrm{d}s$ in which s_{min} is the minimum entropy value in the domain and with determinant $\partial(\hat{\mathsf{x}})/\partial(\mathsf{x})$. The Lagrangian density of the hydrostatic fluid is

$$L = \Theta[s - s_B]\, \sigma \left\{ \left(\tfrac{1}{2} u_m + R_m\right) u^m - U(s, \rho) - g\, z \right\}, \tag{2.40}$$

where $u_m = \delta_{mj} u^j$ is the shorthand defined in (2.9), and $\Theta[\cdot]$ is the Heaviside function, i.e. $\Theta[\phi]$ is zero for $\phi < 0$, one for $\phi > 0$ and half for $\phi = 0$. After some manipulation we find the following relations:

$$\left.\begin{aligned}
(\hat{\Gamma}^{-1})^i_l &= (\Gamma^{-1})^i_l - \frac{\partial\, \delta a^n}{\partial x^j} (\Gamma^{-1})^i_n (\Gamma^{-1})^j_l, \\
\hat{u}^m &= u^m - (\Gamma^{-1})^m_j \frac{\partial\, \delta a^j}{\partial t} - u^j (\Gamma^{-1})^m_n \frac{\partial\, \delta a^n}{\partial x^j}, \\
\sigma(\mathsf{x})\, \hat{U}(\hat{s}, \hat{\rho}) + \sigma(\mathsf{x})\, g\, \hat{z}(\hat{\mathsf{x}}) &= \sigma\, U(s, \rho) + \sigma\, g\, z(\mathsf{x}) + \frac{p}{\rho} (\hat{\sigma} - \sigma) - \frac{\partial\, [p\, (\hat{z} - z)]}{\partial s}
\end{aligned}\right\} \tag{2.41}$$

(all indices are 1 or 2) which prove useful in evaluating (2.39). Using (2.40) and (2.41) (2.39) becomes

$$\begin{aligned}
0 = \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_{min}}^{\infty} \mathrm{d}s\, \Theta[s - s_B(\boldsymbol{x}, t)] \Bigg\{ &-\sigma\, B_M \left[\frac{\partial \ln \sigma_0}{\partial a^k} \delta a^k + (\Gamma^{-1})^j_k \frac{\partial\, \delta a^k}{\partial x^j} \right] \\
&+ \sigma\, (u_m + R_m) \left[(\Gamma^{-1})^m_k \frac{\partial\, \delta a^k}{\partial t} + u^j (\Gamma^{-1})^m_n \frac{\partial\, \delta a^n}{\partial x^j} \right] \Bigg\} \\
- \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_{min}}^{\infty} \mathrm{d}s\, \boldsymbol{\delta}(s &- s_B(\boldsymbol{x}, t))\, \sigma\, B_M\, \delta s_B(\boldsymbol{a}_B)
\end{aligned} \tag{2.42}$$

with B_M given by (2.17). Condition (2.42) implies that

$$\frac{\partial\, \delta a^i(\mathsf{x}, a)|_{\boldsymbol{x},t}}{\partial t} = \frac{\partial\, \delta a^i(\mathsf{x}, a)|_{\boldsymbol{x},t}}{\partial x^j} = 0, \quad \frac{\partial\, [\sigma_0(\boldsymbol{a})\, \delta a^j(\boldsymbol{a})]}{\partial a^j} = 0, \quad \delta a^l_B(\boldsymbol{a}_B) = 0. \tag{2.43}$$

The permitted point variations are thus

$$\delta a^j = -\frac{\epsilon^{jm}}{\sigma_0} \frac{\partial \Psi(\boldsymbol{a})}{\partial a^m} \tag{2.44}$$

with arbitrary function $\Psi(\boldsymbol{a})$ zero at boundaries. Rather than substituting (2.44) into Noether's first theorem, we proceed by analysing (2.42). Further analysis of (2.42) gives the Euler–Lagrange equations for hydrostatic flow and flux terms. Instead of using $\delta a^k(\boldsymbol{x}, t_{0,1}) = 0$ at the endpoints, we find that the relevant flux terms are

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_{min}}^{\infty} \mathrm{d}s \left\{ \frac{\partial}{\partial t} [\Theta[s - s_B(\boldsymbol{x}, t)]\, \sigma\, (u_m + R_m)\, (\Gamma^{-1})_k^m\, \delta a^k] + \frac{\partial}{\partial x^j} [\Theta[s - s_B(\boldsymbol{x}, t)]\, \sigma\, (u_m + R_m)\, (\Gamma^{-1})_k^m\, \delta a^k] \right\} \tag{2.45}$$

while the other flux term cancels as before. Substitution of

$$\delta a^k = -\frac{\epsilon^{kl}}{\sigma_0} (\Gamma^{-1})_l^n \frac{\partial \Psi}{\partial x^n} \tag{2.46}$$

into (2.45) gives

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_{min}}^{\infty} \mathrm{d}s\, \Theta[s - s_B(\boldsymbol{x}, t)] \left(\frac{\partial(\sigma\, q)}{\partial t} + \nabla \cdot (\sigma\, q\, \boldsymbol{v}) \right) \Psi \tag{2.47}$$

(after several integrations by parts and after using boundary conditions) in which the isentropic potential vorticity q (Hoskins 1991) is defined by

$$q = \frac{f + \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{v}}{\sigma}. \tag{2.48}$$

The arbitrariness of Ψ in (2.47) implies flux conservation:

$$\frac{\partial(\sigma\, q)}{\partial t} + \nabla \cdot (\sigma\, q\, \boldsymbol{v}) = 0. \tag{2.49}$$

Combining continuity equation (2.15) and (2.49) yields material conservation of potential vorticity. It follows that besides $\sigma\, q$ also $\sigma\, C(q, s)$, with arbitrary function C, is conserved locally. Globally conserved, so-called Casimir, invariants then have the form

$$\mathscr{C} = \int_{D_H} \int_{s_B}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{d}s\, \sigma\, C(q, s), \tag{2.50}$$

which can be proven directly from the equations of motion (see Appendix A). Without loss of generality Casimir invariants may be split into general and circulation components:

$$\mathscr{C} = \int \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s\, \sigma \left\{ C(q, s) - \lambda(s)\, q \right\} \tag{2.51}$$

in which the last term may be transformed to yield the circulation at the boundaries.

Let us next consider point transformations of the form

$$\hat{t} = t + \delta t(t, \boldsymbol{x}), \quad \hat{\boldsymbol{x}} = \boldsymbol{x}, \quad \hat{s} = s, \quad \hat{a} = a, \tag{2.52}$$

which have to obey (2.39) with L given in (2.40). Such transformations satisfy the following relations:

$$\Delta\left(\frac{\partial a^i}{\partial x^j}\right) = -\frac{\partial\, \delta t}{\partial x^j} \frac{\partial a^i}{\partial t}, \quad \Delta\left(\frac{\partial a^i}{\partial t}\right) = -\frac{\partial\, \delta t}{\partial t} \frac{\partial a^i}{\partial t}, \quad \frac{\partial(\hat{\boldsymbol{x}})}{\partial(\boldsymbol{x})} = \frac{\partial\, \delta t}{\partial t}. \tag{2.53}$$

For transformations (2.52) with (2.53), evaluation of (2.39) yields

$$0=\int_{t_0}^{t_1}\mathrm{d}t\int_{D_H}\mathrm{d}\boldsymbol{x}\int_{s_{min}}^{\infty}\mathrm{d}s\left\{-L\,\frac{\partial\,\delta t}{\partial t}+\Theta\,[s-s_B]\left(B_M\,\sigma_0\,\epsilon_{ij}\epsilon_{kl}\,\frac{\partial a^k}{\partial t}\,\frac{\partial a^l}{\partial x^j}\,\frac{\partial\,\delta t}{\partial x^i}\right.\right.$$
$$\left.\left.-\sigma\,(u_m+R_m)\left[u^l\,(\Gamma^{-1})_j^m\,\frac{\partial\,\delta t}{\partial x^l}\,\frac{\partial a^j}{\partial t}+(\Gamma^{-1})_i^m\,\frac{\partial a^i}{\partial t}\,\frac{\partial\,\delta t}{\partial t}\right]\right)\right\}. \quad (2.54)$$

Hence $\delta t=\epsilon$. First, further evaluation of (2.54) for arbitrary δt yields – cf. (2.35) – the Euler–Lagrange equations and Noether's theorem

$$0=\int_{t_0}^{t_1}\mathrm{d}t\int_{D_H}\mathrm{d}\boldsymbol{x}\int_{s_{min}}^{\infty}\mathrm{d}s\left\{\frac{\partial}{\partial t}\big(\Theta\,[s-s_B]\,E\,\delta t\big)+\frac{\partial}{\partial x^i}\left[\Theta\,[s-s_B]\,\sigma\,u^i\left(\tfrac{1}{2}\,u_m\,u^m+M\right)\delta t\right]\right\} \quad (2.55)$$

with energy density $E\equiv\sigma\,[(1/2)\,u_m\,u^m+U+g\,z]$. Substituting $\delta t=\epsilon$ then yields an energy flux law. Alternatively, let $\delta t=\delta t(t)$ be a function of t only, vanishing at the time to that boundary. A similar calculation to that in Salmon (1983) gives global energy conservation $\mathrm{d}\mathscr{H}/\mathrm{d}t=0$ with Hamiltonian

$$\mathscr{H}=\int_{D_H}\int_{s_B}^{\infty}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\sigma\,\left(\tfrac{1}{2}\,|\boldsymbol{v}|^2+U+g\,z\right). \quad (2.56)$$

A direct verification of energy conservation from the equations of motion is given in Appendix C.

3. Wave-activity invariant and linear stability criteria

A wave-activity conservation law will be derived for the hydrostatic equations of motion by using the energy-Casimir method (e.g. Marsden & Ratiu 1994 and Shepherd 1990). Wave activity is a nonlinearly conserved quantity expressed in terms of disturbances to a basic state. It is second order for small-amplitude disturbances. The conditions for sign definiteness of this small-amplitude wave activity imply non-modal linear stability criteria. Haynes (1988) derived wave-activity conservation relations directly from the hydrostatic equations of motion (in isentropic coordinates) with forcing and dissipation but without inclusion of time-dependent boundaries. In contrast, the energy-Casimir method is used here with explicit inclusion of time-dependent boundaries but without forcing and dissipation.

From the equations of motion (2.15), (2.21), and (2.22) non-resting basic states

$$\boldsymbol{v}=\boldsymbol{U}(\boldsymbol{x}),\ \ \sigma=\Sigma(\boldsymbol{x}),\ \ B=\bar{B}(\boldsymbol{x}),\ \ M=\bar{M}(\boldsymbol{x}),\ \ q=\bar{Q}(\boldsymbol{x}),\ \ \text{and}\ \ s_B=S_B(\boldsymbol{x}),$$

with Bernoulli function $B\equiv(1/2)\,|\boldsymbol{v}|^2+M$, are solutions of the system

$$\left.\begin{aligned}&0=\boldsymbol{U}\cdot\nabla S_B\quad \text{at } s=S_B(\boldsymbol{x}),\\&0=\bar{Q}\,\Sigma\,\hat{\boldsymbol{z}}\times\boldsymbol{U}+\nabla(\tfrac{1}{2}\,|\boldsymbol{U}|^2+\bar{M}),\\&0=\nabla\cdot(\Sigma\,\boldsymbol{U})\end{aligned}\right\} \quad (3.1)$$

and its accompanying diagnostic relations. After introducing a transport streamfunction $\hat{z} \times \nabla \Psi = \Sigma\, \boldsymbol{U}$ it follows that

$$\bar{Q}\, \nabla \Psi = \nabla\, \bar{B} \tag{3.2}$$

and that $\bar{B}, \bar{Q}, S_B$ are constant along streamlines, i.e. are functions of Ψ.

The Casimir function $C(q,s)$ and parameter $\lambda(s)$ in (2.51) may be determined such that the first variation of pseudoenergy $\mathscr{A} = \mathscr{H}[u] + \mathscr{C}[u] - \mathscr{H}[U] - \mathscr{C}[U]$ with state variable $u = \{\boldsymbol{v}, \sigma, s_B\}$ and its basic state $U = \{\boldsymbol{U}, \Sigma, S_B\}$, i.e.

$$\begin{aligned}\delta\mathscr{A} = &\int_{D_H}\int_{s_B}^{\infty} \mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\{(B + C - q\,C_q)\,\delta\,\sigma + (\nabla C_q \times \hat{z} + \sigma\,\boldsymbol{v})\boldsymbol{\cdot}\,\delta\,\boldsymbol{v}\}\\ &-\int_{D_H} \mathrm{d}\boldsymbol{x}\,\{\sigma\,(\bar{B} + C - \lambda\, q)\,\delta\, s_B + (C_q - \lambda)\,\delta\boldsymbol{v} \times \hat{z}\boldsymbol{\cdot}\nabla\, s_B\}|_{s=s_B}\\ &+\int_{\partial D_H} \mathrm{d}l\,\boldsymbol{n}\boldsymbol{\cdot}\int_{s_B}^{\infty}\mathrm{d}s\,(C_q - \lambda)\,\delta\boldsymbol{v}\times\hat{z},\end{aligned} \tag{3.3}$$

vanishes at the basic state. (Variations of Casimir and energy invariants are found in Appendix B.) Hence one finds that

$$\left.\begin{aligned}&\bar{B}(\Psi) = -C(\bar{Q},s) + \bar{Q}\,C_{\bar{Q}}(\bar{Q},s),\\ &\nabla\,C_{\bar{Q}}(\bar{Q},s)\times\hat{z} = -\Sigma\,\boldsymbol{U} = \nabla\Psi\,\times\hat{z},\\ &\lambda(s)|_{(s=S_B,\partial D_H)} = C_{\bar{Q}}(\bar{Q},s)|_{(s=S_B,\partial D_H)}.\end{aligned}\right\} \tag{3.4}$$

Requirement (3.4) may be satisfied at $s = s_B$ since contours of entropy and potential vorticity coincide there for the basic-state flow, and at ∂D_H because no normal flow implies that the basic-state streamfunction there is a function of s only. Alternatively, in a semi-infinite domain potential vorticity may become a function of s only as $|\boldsymbol{x}| \to \infty$.

Defining disturbance quantities in the usual way like $\sigma = \Sigma + \sigma'$, $p = \Pi + p'$, $T = \bar{T} + T'$, etc. one derives the following pseudoenergy:

$$\begin{aligned}\mathscr{A} = &\int_{D_H}\int_{S_B}^{\infty}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\Bigg\{(\Sigma+\sigma')\int_0^{q'}\mathrm{d}\gamma\left[C_\gamma(\bar{Q}+\gamma,s) - C_{\bar{Q}}(\bar{Q})\right]\\ &+\tfrac{1}{2}\,(\Sigma+\sigma')\,|\boldsymbol{v}'|^2 + \sigma'\,\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{v}'\Bigg\} + \int_{D_H}\int_{S_B+s'_B}^{\infty}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,(\Sigma+\sigma')\,E(\Sigma+\sigma')\\ &-\int_{D_H}\int_{S_B}^{\infty}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\{\Sigma\,E(\Sigma,s) + \bar{M}\,\sigma'\}\\ &+\int_{D_H}\mathrm{d}\boldsymbol{x}\,\{(p\,z)\big|_{S_B+s'_B} - \Pi_B\,Z_B + \Sigma_B\,g\,Z_B\,s'_B\,\}\\ &-\int_{D_H}\int_{S_B}^{S_B+s'_B}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\left\{\tfrac{1}{2}\,\Sigma\,|\boldsymbol{U}|^2 + \Sigma_B\,g\,Z_B + \Sigma\,C(\bar{Q},s) - \lambda(s)\,\bar{Q}\,\Sigma\right\}\\ &-\int\int_{S_B}^{S_B+s'_B}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\{(\Sigma+\sigma')\left[\,\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{v}' + \tfrac{1}{2}\,|\boldsymbol{v}'|^2 + \Sigma\,C(Q+q',s) - \lambda(s)\,q'\right]\\ &+\tfrac{1}{2}\,\sigma'\,|\boldsymbol{U}|^2 - \Sigma\,C(\bar{Q},s) - \lambda(s)\,\bar{Q}\,\sigma'\},\end{aligned} \tag{3.5}$$

in which the subscript in Π_B denotes evaluation of Π at the steady-state boundary $s = S_B$ and so forth except in s'_B. By construction, we have $\mathrm{d}\mathscr{A}/\mathrm{d}t = 0$. For simplicity we first consider only the small-amplitude limit $\mathscr{A}_2$ of (3.5) for which the boundary coincides with an isentrope s_0. The result is a quantity of second order in the disturbance amplitude

$$\mathscr{A}_2 = \int_{D_H} \int_{s_0}^{\infty} \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \left\{ \frac{1}{2\,(\Sigma + \sigma')} \left| (\Sigma + \sigma')\,\boldsymbol{v}' + \boldsymbol{U}\,\sigma' \right|^2 - \frac{1}{2}\,\frac{|\boldsymbol{U}|^2}{(\Sigma + \sigma')}\,\sigma'^2 + \frac{1}{\Sigma}\,\frac{\partial Z}{\partial s}\,p'\,\sigma' + \frac{1}{2}\left(\frac{R}{c_p} - 1\right)\frac{1}{\Pi}\,\frac{\partial Z}{\partial s}\,p'^2 + \tfrac{1}{2}\,C''(\bar{Q})\,q'^2 \right\} \quad (3.6)$$

(use has been made of the ideal gas law). The appearance of terms proportional to $\sigma'\,p'$ and p'^2 in (3.6) prevents the derivation of linear or formal stability criteria for general moving basic-states. Holm & Long (1989) derived formal stability criteria for hydrostatic, incompressible Boussinesq flows expressed in isopycnal coordinates by introducing an effective local wavenumber, which in the isentropic coordinates used here would amount to σ'/p', but their criteria are conditional in that they are dependent on the nature of the perturbations rather than only on the nature of the steady state.

For a resting basic state

$$C(s) = -\bar{M}(s) = -\int_{s_B}^{s} \mathrm{d}\gamma \left(\frac{\Pi(\gamma)}{p_{00}}\right)^{c_p/R} \mathrm{e}^{-(\gamma - s_{00})/c_p}, \quad (3.7)$$

cf. expression (8.9) in Shepherd (1993). The small-amplitude limit of (3.5) for a resting basic state is a second-order quantity $\mathscr{A}_R$ of the form

$$\mathscr{A}_R = \frac{1}{2}\int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_0}^{\infty} \mathrm{d}s \left\{ \Sigma\,|\boldsymbol{v}|^2 + \frac{1}{g\,c_p\bar{\rho}}\,p'^2 \right\} + \frac{1}{2}\int_{D_H} \mathrm{d}\boldsymbol{x}\,g \left\{ \Sigma\,\frac{\partial Z}{\partial s}\,[s'_B - \frac{p'}{g\,\Sigma}]^2 \right\}\bigg|_{S_B}. \quad (3.8)$$

This expression may also be derived directly from the equations of motion linearized around a resting basic state – see Appendix D – except that the boundary contribution is then absent. No contradiction ensues because the time derivatives of the integrands in the boundary integral are zero:

$$\left[\frac{\partial s_B}{\partial t} - \frac{1}{g\,\Sigma}\,\frac{\partial p'}{\partial t}\right]\bigg|_{S_B} = \left[\frac{\partial s_B}{\partial t} + \left(\frac{\partial Z}{\partial s}\right)^{-1}\frac{\partial z'}{\partial t}\right]\bigg|_{S_B} = 0. \quad (3.9)$$

Linear stability criteria for a resting basic state follow from (3.8) as

$$\frac{1}{g\,c_p}\,\frac{1}{\bar{\rho}} > 0, \quad \Sigma = \bar{\rho}\,\frac{g}{c_p}\,N^{-2} > 0, \quad \left(\Sigma\,\frac{\partial Z}{\partial s}\right)\bigg|_{S_B} > 0. \quad (3.10)$$

The first condition in (3.10) ensures the natural positive basic-state density and the second one then ensures static stability.

4. Discussion of conservative approximations

When one considers the hydrostatic equations after they are linearized around a state of rest, then one finds in mid-latitudes and away from lateral boundaries that all gravity modes have frequencies larger than f. (This follows for example readily for an isothermal basic state in a domain that is bounded by two vertical planes and that is horizontally periodic with constant f.)

The time-scale separation between linear gravity and vortical modes in mid-latitudes suggests that approximate models which only describe the large-scale vortical motion may be valid even in the nonlinear limit. This is the case as long as the generation of gravity-wave motion by vortical motion remains weak.

When the aim is to model such large-scale rapidly-rotating geophysical flows, one advantage of a variational or Hamiltonian formulation is that we may approximate these formulations such as to preserve analogues of symmetries in the original equations of motion. Analogues of conservation laws in the original parent dynamics can then be preserved in situations where we desire that the approximate dynamics is conservative in an appropriate inviscid limit. The subsequent outline will show how approximations can be made to the hydrostatic isentropic variational formulation derived in this paper. Gravity modes in these so-called 'balanced models' are filtered. They disappear in these approximations because the time order of the system has been reduced in a perturbation or truncation approach, which is called singular in the sense used in perturbation methods (e.g. Bender & Orszag 1978).

In the geophysical fluid dynamics literature, variational principles for shallow-water and stratified Boussinesq equations have been used as starting point or as parent dynamics for deriving approximate balanced models (Salmon 1983, 1985, 1988*b*; Allen & Holm 1996; Holm 1996; McIntyre & Roulstone 1996). Several important points arise. First, Eulerian variational principles such as (2.32) may be scaled using Eulerian estimates of time, length and velocity scales, as opposed to scaling a Lagrangian variational principle. Secondly, direct substitution of velocity constraints $\boldsymbol{v} \equiv \boldsymbol{v}^C[\sigma, s_B]$ into (2.32) expressed in terms of the pseudodensity σ and s_B, either obtained by asymptotic means or physical intuition, yields a Hamilton's principle in which variations are taken with respect to fluid label variables $\boldsymbol{a}$ only. The variations now yield two first-order partial differential equations in time, one for each a^k, as opposed to the two second-order partial differential equations in time arising in the original hydrostatic parent dynamics. This again underscores the reduction of order in time. The direct substitution approach carries over directly from the shallow-water realm for the interior dynamics since the hydrostatic system has, on each isentropic surface, a structure quite similar to that of the shallow-water system. That is, if we replace Montgomery potential M by shallow-water depth h and pseudodensity σ by h as well, then the shallow-water equations emerge from (2.15) and (2.22). Similarly, the shallow-water Eulerian Hamilton's or action principle lacks reference (and integration over) entropy s, and its potential energy $(1/2)\,h^2$ replaces the hydrostatic internal and potential energy. Thirdly, the velocity $\boldsymbol{v}$ resulting from the direct substitution approach differs from the constraint velocity $\boldsymbol{v}^C[\sigma, s_B]$ and is the 'particle' velocity (McIntyre & Roulstone 1996) $u_m = u_m^P \equiv -\delta_{mj}(\Gamma^{-1})^j_k\, \partial a^k/\partial t$ that advects the fluid particles. (McIntyre & Roulstone 1996 refer to the appearance of a particle velocity and constrained velocity as 'velocity splitting'.) Under certain restrictions there exists a unique solution of an elliptic set of equations for $\boldsymbol{v}^P$. Finally, the boundary dynamics in these approximate or balanced models are similar to those in the full hydrostatic system; the particle velocity at the boundary advects the boundary entropy.

Let us introduce the anisotropic scaling often used in large-scale geophysical fluid dynamics on frontal ($\leqslant 100\,$km) and synoptic scales ($O(1000\,$km$)$)

$$\left.\begin{aligned} &x = l\,x', \quad y = L\,y', \quad z = D\,z', \quad u = U\,u', \quad v = V\,v', \\ &a = A\,a', \quad b = B\,b', \quad t = (l/U)\,t', \quad M = \mu\,M', \quad f = f_0\,f', \quad s = s_{00}\,s', \end{aligned}\right\} \quad (4.1)$$

with $(l/U) = (L/V)$. We take $\mu = f_0\, U\, l$ to enforce geostrophic balance, that is

$$f\,\hat{z} \times \boldsymbol{v} = -\nabla M, \tag{4.2}$$

at leading order in at least the along-frontal y-direction. Analysing the magnitude of the terms $(\Gamma^{-1})^j_k\, \partial a^k/\partial t$ in (2.32) reveals that they scale as the velocity; hence the reference to label scales A and B does not appear explicitly. After dropping the primes, a scaled version of (2.32) is

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s \left\{ -\sigma \left(Ro\, \delta^2\, u_1 + R_1 \right) \left(\Gamma^{-1} \right)^1_k \frac{\partial a^k}{\partial t} - \sigma \left(Ro\, u_2 + R_2 \right) \left(\Gamma^{-1} \right)^2_k \frac{\partial a^k}{\partial t} \right.$$
$$\left. -\sigma \left(\tfrac{1}{2}\, Ro\, \delta^2\, u^2 + \tfrac{1}{2}\, Ro\, v^2 + \frac{U(\rho, s)}{f_0\, U\, L} + \frac{g\, D}{f_0\, U\, L}\, z \right) \right\} \tag{4.3}$$

with the following dimensionless parameters: the anisotropic Rossby number $Ro \equiv v/(f_0\, l)$ and frontal parameter $\delta \equiv l/L = U/V$. Proper scaling of the internal energy and potential energy requires the introduction of a basic state, e.g. the basic state used in § 3 which depends on entropy only. If we take $\delta = 1$ then the quasi-geostrophic system (e.g. Pedlosky 1987) arises at leading order in a Rossby-number expansion and this quasi-geostrophic dynamics is expected to be valid only in the neighbourhood of the basic state around which the expansion is ordered. This restriction can be lifted in a rather *ad hoc* manner by assuming that the magnitude of the total internal and potential energy is of order unity. Consequently, a division between basic state and remainder is absent. Scaling is thus only used to compare relative magnitudes of the kinetic and Coriolis terms in the variational principle.

The scaled action principle (4.3) covers various small-parameter limits. Several leading- and higher-order balanced models arise in these limits:

(i) For $\delta = 0$ and $Ro = O(1)$ an isentropic version of Hoskins & Bretherton's (1972) geostrophic momentum model appears.

(ii) For $\delta = O(1)$ and $Ro = 0$ geostrophic balance (4.2) appears.

(iii) Substituting geostrophic velocity $\boldsymbol{v}^G \equiv (1/f)\hat{z} \times \nabla M$ as constraint velocity $\boldsymbol{v}^C = \boldsymbol{v}^G$ back into (4.3) yields an isentropic version – instead of a shallow-water version – of Salmon's L1-dynamics (Salmon 1985). In shallow-water L1-dynamics the particle velocity is a unique solution of a system of elliptic equations provided that potential vorticity q is positive and the domain bounded (Ren & Shepherd 1997). This result carries over to hydrostatic L1-dynamics, at least for constant s_B and in a domain bounded in the horizontal, provided that potential vorticity $q > 0$ on each isentropic surface.

(iv) Higher-order ageostrophic balanced models may now in principle be derived by substituting the L1-dynamics particle velocity constraint back into (4.3), and so forth. Although this can be made plausible by performing the variations in (4.3) for arbitrary constraint velocity, the well-posedness of the resulting systems needs further investigation.

The above program (i)–(iv) concerns research quite similar to previous research done in the similar shallow-water system, but would extend it by including stratification and boundary intersecting isentropes. Finding an Eulerian variational or Hamiltonian formulation of the hydrostatic equations originated in the quest for such a formulation for (an isentropic version of) the Hoskins & Bretherton geostrophic momentum equations. Here, I will only derive an isentropic version of their model from the

Eulerian variational principle in detail. For $\delta = 0$ the dimensional form of (4.3) becomes

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s \Big\{ -\sigma\, R_1 \left(\Gamma^{-1}\right)^1_k \frac{\partial a^k}{\partial t}$$

$$-\sigma \left(u_2 + R_2\right) \left(\Gamma^{-1}\right)^2_k \frac{\partial a^k}{\partial t} - \sigma\, \left(\tfrac{1}{2} v^2 + U(\rho, s) + g\, z\right) \Big\}. \tag{4.4}$$

Variations in δa and δb yield geostrophic balance along the front and the cross-frontal momentum equation

$$f\, v = \frac{\partial M}{\partial x}, \tag{4.5}$$

$$\frac{\partial v}{\partial t} + \boldsymbol{v} \cdot \nabla v + f\, u = -\frac{\partial M}{\partial y}, \tag{4.6}$$

respectively, and the variation in δv yields

$$(\Gamma^{-1})^2_k \frac{\partial a^k}{\partial t} + v = 0 \tag{4.7}$$

with $\boldsymbol{v} = (u, v)^T \equiv \boldsymbol{v}^P$ and $u^C = 0$. The zonal velocity u is understood to be a shorthand for

$$-(\Gamma^{-1})^1_k \frac{\partial a^k}{\partial t} \tag{4.8}$$

as it was in the Eulerian Hamilton's principle (2.10). The meridional velocity v is not constrained *a priori* in the variational principle but is geostrophically balanced by (4.5). The full continuity equation $\partial\sigma/\partial t + \nabla\cdot(\sigma\, \boldsymbol{v}) = 0$ is retained from the definition of σ in (2.11) and the definition of particle velocity in (4.7) and (4.8). By using Noether's theorem, as in § 2.4, we again find potential vorticity and energy conservation but now with potential vorticity

$$q_{HB} \equiv \frac{v_x + f}{\sigma}, \tag{4.9}$$

and with Hamiltonian

$$\mathscr{H}_{HB} \equiv \int_{D_H} \int_{s_B}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{d}s\, \sigma\, \left\{ \tfrac{1}{2} v^2 + U(s, \rho) + g\, z \right\}. \tag{4.10}$$

These simplifications directly follow from a substitution $u = u_C = 0$ into the parent action principle (4.3).

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Appendix A. Casimir invariants

The invariance of Casimir (2.50), i.e.

$$\mathscr{C} = \int_{D_H} \int_{s_B}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{d}s\, \sigma\, C(q, s), \tag{A 1}$$

may be shown directly from the equations of motion. That is

$$
\begin{aligned}
\frac{\mathrm{d}\mathscr{C}}{\mathrm{d}t} &= \int \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s \left(\frac{\partial \sigma}{\partial t} C + \sigma\, C_q\, \frac{\partial q}{\partial t} \right) - \int \mathrm{d}\boldsymbol{x}\, (\sigma\, C)|_{s_B}\, \frac{\partial s_B}{\partial t} \\
&= - \int \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s \nabla \cdot (\sigma\, \boldsymbol{v}\, C) + \int \mathrm{d}\boldsymbol{x}\, (\sigma\, C\, \boldsymbol{v})|_{s_B} \cdot \nabla s_B \\
&= - \int \mathrm{d}\boldsymbol{x}\, \nabla \cdot \int_{s_B}^{\infty} \mathrm{d}s\, \sigma\, \boldsymbol{v}\, C \\
&= - \int_{\partial D_H} \mathrm{d}l\, \boldsymbol{n} \cdot \int_{s_B}^{\infty} \mathrm{d}s \left(\sigma\, \boldsymbol{v}\, C \right)_{s_B} = 0,
\end{aligned}
\qquad \text{(A 2)}
$$

where $\mathrm{d}l$ is an infinitesimal line element at the extreme horizontal limits ∂D_H with $C_q \equiv \partial C / \partial q$. The last boundary contribution is seen to cancel when there are walls, either because at vertical boundaries $\hat{\boldsymbol{n}} \cdot \boldsymbol{v} = 0$ or because the integral over s has identical limits when there are non-vertical boundaries, or for vanishing or cancelling flows at infinity.

Appendix B. Variations of Casimir and energy invariants

Variations of Casimir and energy invariants are used in § 3. Variation of Casimir invariant (2.51) gives

$$
\begin{aligned}
\delta \mathscr{C} = & \int \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s \left\{ \left(C - q\, C_q \right) \delta \sigma + \nabla C_q \times \hat{\boldsymbol{z}} \cdot \delta \boldsymbol{v} \right\} \\
& + \int_{\partial D_H} \mathrm{d}l \int_{s_B}^{\infty} \mathrm{d}s\, \hat{\boldsymbol{n}} \cdot \left(C_q - \lambda \right) \delta \boldsymbol{v} \times \hat{\boldsymbol{z}} \\
& + \int \mathrm{d}\boldsymbol{x} \left\{ \left(C_q - \lambda \right) \delta \boldsymbol{v} \times \hat{\boldsymbol{z}} \cdot \nabla s_B - \sigma \left(C - \lambda\, q \right) \delta s_B \right\} \Big|_{s = s_B}.
\end{aligned}
\qquad \text{(B 1)}
$$

Variation of the Hamiltonian invariant gives

$$
\begin{aligned}
\mathscr{H} = & \int_{D_H} \int_{s_B}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{d}s \left\{ \left(\tfrac{1}{2}\, |\boldsymbol{v}|^2 + M \right) \delta \sigma + \sigma\, \boldsymbol{v} \cdot \delta \boldsymbol{v} \right\} \\
& - \int_{D_H} \int_{s_B}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{d}s\, \sigma \left(\tfrac{1}{2}\, |\boldsymbol{v}|^2 + M \right) \Big|_{s = s_B} \delta s_B.
\end{aligned}
\qquad \text{(B 2)}
$$

Appendix C. Direct calculation of energy conservation

In this Appendix energy conservation will be derived directly from the equations of motion

$$
\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + f\, \hat{\boldsymbol{z}} \times \boldsymbol{v} = - \nabla M, \quad \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma\, \boldsymbol{v}) = 0 \qquad \text{(C 1}a,b\text{)}
$$

with

$$
\sigma = - \frac{p_{00}}{g} \frac{\partial}{\partial s} \left[\left(\frac{1}{T_{00}} \frac{\partial M}{\partial s} \right)^{c_p / R} \mathrm{e}^{-(s - s_{00})/R} \right], \qquad \text{(C 2)}
$$

the first law of thermodynamics (2.3) and definitions

$$
M = E + g\, z, \quad E = U + p/\rho, \quad g\, \sigma = \rho\, \frac{\partial (g\, z)}{\partial s} = - \frac{\partial p}{\partial s}. \qquad \text{(C 3)}
$$

Multiplication of (C 1a) by $\sigma \boldsymbol{v}$, (C 1b) by $(1/2)\,|\boldsymbol{v}|^2$, (C 1$b$) by M, and summation gives

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\,\sigma\,|\boldsymbol{v}|^2\right) + \nabla\cdot\left[\sigma\,\boldsymbol{v}\,\left(\tfrac{1}{2}|\boldsymbol{v}|^2 + M\right)\right] + M\,\frac{\partial \sigma}{\partial t} = 0. \tag{C 4}$$

With definitions (C 3) and the first law of thermodynamics, the last term in (C 4) decomposes into

$$M\,\frac{\partial \sigma}{\partial t} = \frac{\partial[\sigma\,(U + g\,z)]}{\partial t} + \frac{\partial}{\partial s}\left(p\,\frac{\partial z}{\partial t}\right). \tag{C 5}$$

By combining (C 4) and (C 5), integrating the result over the domain (e.g. take $s = s_B(\boldsymbol{x}, t), \ldots, \infty$), applying suitable boundary conditions for inviscid flow, and using $\partial s_B/\partial t + \boldsymbol{v}\cdot\nabla s_B = 0$ at the lower boundary $z = h_B$ or $s = s_B$, we find

$$\frac{\partial}{\partial t}\int_{D_H} \mathrm{d}\boldsymbol{x} \int_{s_B}^{\infty} \mathrm{d}s\,\sigma\,\left(\tfrac{1}{2}\,|\boldsymbol{v}|^2 + (U + g\,z)\right) - \int_{D_H} \mathrm{d}\boldsymbol{x}\left(p\,\frac{\partial z}{\partial t} + p\,\frac{\partial z}{\partial s}\,\frac{\partial s_B}{\partial t}\right)\bigg|_{s_B}. \tag{C 6}$$

These last two boundary terms cancel one another and hence energy conservation is established by direct calculation.

Appendix D. Linear stability

In this Appendix linear stability criteria will be derived from the equations of motion linearized around a resting basic state with pseudodensity

$$\Sigma(s) = -\frac{p_{00}}{g}\,\frac{\partial}{\partial s}\left[\left(\frac{1}{T_{00}}\,\frac{\partial \bar{M}}{\partial s}\right)^{c_p/R} \mathrm{e}^{-(s-s_{00})/R}\right], \tag{D 1}$$

density

$$\bar{\rho} = \frac{p_{00}}{T_{00}\,R}\left(\frac{1}{T_{00}}\,\frac{\partial \bar{M}}{\partial s}\right)^{(c_p/R)-1} \mathrm{e}^{-(s-s_{00})/R}, \tag{D 2}$$

and Montgomery potential $\bar{M}(s)$. Introducing perturbation variables in the usual way, $M = \bar{M} + m, \sigma = \Sigma + \sigma', s_B = S_B + s'_B$, etc. the linearized equations read

$$\frac{\partial \boldsymbol{v}}{\partial t} + f\,\hat{\boldsymbol{z}} \times \boldsymbol{v} = -\nabla m, \quad \frac{\partial \sigma'}{\partial t} + \Sigma(s)\,\nabla\cdot\boldsymbol{v} = 0 \tag{D 3a,b}$$

with

$$\sigma' = -\frac{c_p\,p_{00}}{g\,T_{00}\,R}\frac{\partial}{\partial s}\left[\left(\frac{1}{T_{00}}\,\frac{\partial \bar{M}}{\partial s}\right)^{(c_p/R-1)}\frac{\partial m}{\partial s}\,\mathrm{e}^{-(s-s_{00})/R}\right], \tag{D 4}$$

using the first law of thermodynamics and definitions (C 3).

Multiplication of (D 3a) by $\Sigma\,\boldsymbol{v}$, (D 3b) by m, and addition gives

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\,\Sigma\,|\boldsymbol{v}|^2\right) + \nabla\cdot(\Sigma\,m\,\boldsymbol{v}) + m\,\frac{\partial \sigma'}{\partial t} = 0. \tag{D 5}$$

Using (D 2), (D 4) and $p' = c_p\,\bar{\rho}\,\partial m/\partial s$ the last term integrated over entropy s (e.g. from $s = S_B, \ldots, \infty$) gives

$$\int_{S_B} \mathrm{d}s\; m\,\frac{\partial \sigma'}{\partial t} = \frac{m}{g}\,\frac{\partial p'}{\partial t}\bigg|_{s=S_B} + \int_{S_B} \mathrm{d}s\;\frac{1}{g\,c_P\,\bar{\rho}}\,\frac{\partial}{\partial t}\left(\tfrac{1}{2}\,p'^2\right). \tag{D 6}$$

Integration of (D 5) over the domain, using result (D 6), careful integration by parts and application of suitable boundary conditions for inviscid flows, eventually gives

$$\frac{\partial}{\partial t}\int_{D_H}\mathrm{d}\boldsymbol{x}\int_{S_B}^{\infty}\mathrm{d}s\,\tfrac{1}{2}\left(\Sigma|\boldsymbol{v}|^2+\frac{1}{g\,c_P\,\bar{\rho}}\,p'^2\right)+\int_{D_H}\mathrm{d}\boldsymbol{x}\,m\left(\frac{1}{g}\frac{\partial p'}{\partial t}-\Sigma\,\frac{\partial s_B}{\partial t}\right)\bigg|_{s=S_B}=0. \quad \text{(D 7)}$$

The last two terms cancel one another after using the linearized hydrostatic equation and the linearized (2.20). Hence, linear stability criteria (3.10) have also been derived by direct manipulation of the linearized equations of motion.

REFERENCES

Allen, J. S. & Holm, D. D. 1996 Extended-geostrophic Hamiltonian models for rotating shallow water motion. *Physica* D **98**, 229–248.

Bender, C. M. & Orszag, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill.

Bokhove, O. 1996 On balanced models in geophysical fluid dynamics: slowest invariant manifolds, slaving principles, and Hamiltonian structure. PhD Dissertation, University of Toronto, 186 pp.

Bokhove, O. 1999 Balanced models in geophysical fluid dynamics: Hamiltonian formulation, constraints and formal stability. In *Mathematics of Atmosphere and Ocean Dynamics, Proc. Isaac Newton Institute of Mathematical Sciences Programme July–Dec. 1996* (ed. J. Norbury & I. Roulstone). Cambridge University Press (to appear).

Dirac, P. A. M. 1964 *Lectures on Quantum Mechanics.* Belfer Graduate School of Science, Yeshiva University, New York.

Haynes, P. H. 1988 Forced, dissipative generalizations of finite-amplitude wave-activity conservation relations for zonal and non-zonal basic flows. **45**, 2352–2362.

Holm, D. D. 1996 Hamiltonian balance equations. *Physica* D **98**, 379–414.

Holm, D. D. & Long, B. 1989 Liapunov stability of ideal stratified fluid equilibria in hydrostatic balance. *Nonlinearity* **2**, 23–35.

Hoskins, B. J. 1991 Towards a PV-θ view of the general circulation. *Tellus* **43AB**, 27–35.

Hoskins, B. J. & Bretherton, F. P. 1972 Atmospheric frontogenesis models: mathematical formulation and solution. *J. Atmos. Sci.* **29**, 11–37.

Hoskins, B. J., McIntyre, M. E. & Robertson, A. W. 1985 On the use and significance of isentropic potential vorticity maps. *Q. J. R. Met. Soc.* **111**, 877–946.

Marsden, J. E. & Ratiu, T. S. 1994 *Introduction to Mechanics and Symmetry*. Springer.

Marsden, J. E. & Weinstein, A. 1983 Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. *Physica* D **4**, 394–406.

Mazer, A. & Ratiu, T. S. 1989 Hamiltonian formulation of adiabatic free boundary flows. *J. Geomet. Phys.* **6**, 271–291.

McIntyre, M. E. & Roulstone, I. 1996 Hamiltonian balanced models: constraints, slow manifolds and velocity splitting. Submitted to *J. Fluid Mech.*

Morrison, P. J. 1998 Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.* **70**, 467–521.

Noether, E. 1918 Invariante variationsprobleme. *Kgl. Ges. Wiss. Nachr. Göttingen Math. Physik.* **2**, 235–257.

Padheye, N. & Morrison, P. J. 1996 Relabeling symmetries in hydrodynamics and magnetohydrodynamics. *Plasma Phys. Rep.* **22**, 869–877.

Pedlosky, J. 1987 *Geophysical Fluid Dynamics.* Springer.

Ren, S. & Shepherd, T. G. 1997 Lateral boundary contributions to wave-activity invariants and nonlinear stability theorems for balanced dynamics. *J. Fluid Mech.* **345**, 287–305.

Roulstone, I. & Brice, S. J. 1995 A Hamiltonian formulation of the quasi-hydrostatic equations. *Q. J. R. Met. Soc.* **121**, 927–936.

Salmon, R. 1983 Practical use of Hamilton's principle. *J. Fluid Mech.* **132**, 431–444.

Salmon, R. 1985 New equations for nearly geostrophic flow. *J. Fluid Mech.* **153**, 461–477.

Salmon, R. 1988*a* Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.* **20**, 225–256.

Salmon, R. 1988*b* Semigeostrophic theory as a Dirac-bracket projection. *J. Fluid Mech.* **196**, 345–358.

Shepherd, T. G. 1990 Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics. *Adv. Geophys.* **32**, 287–338.

Shepherd, T. G. 1993 A unified theory of available potential energy. *Atmos. Ocean* **31**, 1–26.

Sudarshan, E. C. G. & Mukunda N. 1974 *Classical Dynamics: a Modern Perspective*. John Wiley.

Theiss, J. 1997 *Hamiltonian Balanced Models*. Essay for the J. T. Knight Prize, University of Cambridge, 37 pp.